Solution 10

Supplementary Problems

1. Let D be the parallelogram formed by the lines $x+y=1, x+y=3, y=2x-3, y=2x+2$. Evaluate the line integral
$$\oint_C dx + 3xy\, dy$$
where C is the boundary of D oriented in anticlockwise direction. Suggestion: Try Green's theorem and then apply change of variables formula.

Solution. By Green's theorem
$$\oint_C dx + 3xy\, dy = \iint_D 3y\, dA(x,y)\ .$$
Next, let $u = x+y$ and $v = y-2x$. Then $(u,v) \mapsto (x,y)$ sends the rectangle $R = [1,3]\times[-3,2]$ to D. We have $\dfrac{\partial(u,v)}{\partial(x,y)} = 3$ and $x = (u-v)/3$ and $y = (2u+v)/3$. By the change of variables formula
$$\begin{aligned}\iint_D 3y dA(x,y) &= \iint_R (2u+v)\frac{1}{3}\, dA(u,v) \\ &= \frac{1}{3}\int_1^3\int_{-3}^2 (2u+v)\, dvdu \\ &= \frac{1}{3}\int_1^3 (10u-5)\, du \\ &= \frac{35}{3}\ .\end{aligned}$$

2. Let $F = M\mathbf{i} + N\mathbf{j}$ be a smooth vector field in $\mathbb{R}^2$ except at the origin. Suppose that $M_y = N_x$. Show that for any simple closed curve γ enclosing the origin and oriented in anticlockwise direction, one has
$$\oint_\gamma M dx + N dy = \varepsilon \int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta]\, d\theta\ ,$$
for all sufficiently small ε. What happens when γ does not enclose the origin?

Solution. Let γ_ε be the circle entered at the origin with radius ε which is so small to be enclosed by γ. Then the vector field $\mathbf{F}$ is smooth in the region bounded by γ and γ_1. Applying Green's theorem in a multi-connected region we have
$$\oint_\gamma M dx + N dy = \oint_{\gamma'} M dx + N dy\ .$$
Using the standard parametrization, $\theta \mapsto (\varepsilon\cos\theta, \varepsilon\sin\theta)$, we further have
$$\oint_{\gamma'} M dx + N dy \ = \varepsilon \int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta]\, d\theta\ ,$$
for all sufficiently small ε.

The line integral vanishes when γ does not include the origin.

3. (15 points) Let

$$\mathbf{H} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j},$$

which is defined in the plane except at the origin.

(a) Explain why $\mathbf{H}$ is conservative in the upper half plane $\{(x,y):\ y>0\}$.

(b) Find a potential function for $\mathbf{H}$ in the upper half plane.

Solution. (a) For the vector field $\mathbf{H}$, we have

$$M_y = \frac{y^2-x^2}{(x^2+y^2)^2} = N_x\ ,$$

hence the component test is fulfilled. Since the upper half plane is simply-connected, it implies that $\mathbf{H}$ admits a potential function.

(b) As one can verify directly, a potential function is given by $\arctan\frac{y}{x}$.

Note. Indeed, the function $\arctan\frac{y}{x}$ is a potential function for $\mathbf{H}$ in the region $\{(x,y) : (x,y) \neq (x,0), x \leq 0\}$, that is, the plane minus the non-positive x-axis. This is a simply connected region.